Some Aspects of Harmonic Analysis of Data Gridded on the Ellipsoid

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Abstract. In producing the next Earth Gravity Model (EGM), one task is to compute the spherical harmonic spectrum of gravity anomalies for which a global grid of band-limited area-means has been computed on the ellipsoid. For EGM96, a solid spherical harmonic formulation was used to compute the spectrum to degree 359, directly from 30′×30′ gridded anomalies using block-diagonal least squares [BDLS]. For the new EGM, an identical approach would use 5′×5′ gridded anomalies and BDLS to degree 2159. In the latter case, this BDLS approach exhibits minor instabilities in recovering zonal harmonics. Contrary to expectation, these instabilities result from converting gridded equiangular geodetic latitudes to non-equiangular geocentric latitudes, as this increases the non-orthogonality of the discretized Legendre polynomials and so causes the recovered zonal coefficients to be almost totally correlated. An alternative is to first compute the ellipsoidal harmonic spectrum to degree 2159, since the non-equiangular spacing of the reduced latitudes does not correlate the recovered zonal ellipsoidal harmonics. *Jekeli's* (1988) transformation is then used to transform the ellipsoidal harmonic spectrum to spherical harmonics. Importantly, the final spherical spectrum must be truncated above degree 2159, otherwise the omission errors at polar latitudes become greatly amplified when continued down to the ellipsoid.

Keywords. Earth Gravity Model, Spherical Harmonics, Ellipsoidal Harmonics, Block Diagonal Least Squares

1 Introduction

The National Geospatial-Intelligence Agency [NGA] is currently sponsoring the development of an Earth Gravitational Model [EGM], complete to (at least) spherical harmonic degree and order 2160. This model is intended to replace EGM96 (*Lemoine et al.*, 1998), which was complete to degree and order 360. The development of EGM96 included the analysis of a global grid of 30′×30′ gravity anomaly area-means, equiangular in geodetic latitude. This process is described in detail by *Pavlis* in (*Lemoine et al.*, 1998, Chap. 8). In brief, starting from 30′×30′ area-mean values of Molodensky free-air gravity anomalies defined on the topography, one applies ellipsoidal corrections, downward continuation, and a spectral filtering technique in order to reduce these anomalies, as closely as possible, to quantities $\overline{\Delta g}_{ij}^{e}$ that are band-limited to degree 359, refer to the surface of the ellipsoid, and are related to the potential coefficients by:

$$\overline{\Delta g}_{ij}^{e} = \overline{\Delta g}(r_i^e, \theta_i, \lambda_j) = \\ = \frac{1}{\Delta\sigma_i} \frac{GM}{(r_i^e)^2} \sum_{n=0}^{N} (n-1) \left(\frac{a}{r_i^e}\right)^n \sum_{m=-n}^{n} \bar{C}_{nm} \cdot \overline{IY}_{nm}^{ij} , \tag{1}$$

with the area element:

$$\Delta\sigma_i = \Delta\lambda \int_{\theta_i}^{\theta_{i+1}} \sin\theta d\theta = \Delta\lambda \cdot \left(\cos\theta_i - \cos\theta_{i+1}\right), \tag{2}$$

and the integrated surface spherical harmonic:

$$\overline{IY}_{nm}^{ij} = \int_{\theta_i}^{\theta_{i+1}} \bar{P}_{n|m|}(\cos\theta) \sin\theta d\theta \times \\ \int_{\lambda_j}^{\lambda_{j+1}} \begin{Bmatrix} \cos m\lambda \\ \sin|m|\lambda \end{Bmatrix} d\lambda \quad \begin{matrix} \text{if } m \geq 0 \\ \text{if } m < 0 \end{matrix} . \tag{3}$$

r_i^e is geocentric distance, θ_i co-latitude, and λ_j longitude, associated with a cell on the *i*-th "row" and *j*-th "column" of a global equiangular grid. N denotes the maximum degree (n) and order (m). $\bar{C}_{nm}$ are fully-normalized coefficients of the disturbing potential and $\bar{P}_{n|m|}$ are fully-normalized Associated Legendre functions. *GM* is the geocentric gravitational constant and a is a scale factor associated with the coefficients $\bar{C}_{nm}$ (usually numerically equal to the semi-major axis of the reference ellipsoid). Specific properties of the reference surface and the geometry of the data grid, as well as the error properties of the data, cause the normal matrix resulting from the ensemble of observation equations (1) to be of block-diagonal nature (*Colombo*, 1981; *Pavlis*, 1988). This aspect was exploited during the development of EGM96, where a Block-Diagonal Least-Squares adjustment technique [BDLS] was used to estimate the portion of the model from degree 71 to degree and order 359, from the combination of a satellite-only model with terrestrial data. *Pavlis* in (*Lemoine et al.*, 1998, Sect. 8.2.4) describes in detail this adjustment. The following discussion focuses *only* on the analysis of the terrestrial data, using the *solid* spherical harmonic formulation of equation (1). Here, we restrict ourselves to a "Type 1" block-diagonal structure (*ibid.*, Sect. 8.2.2), and to gravity anomalies with equal weights, since this is sufficient within the context of the present study.

This Type 1 block-diagonal structure implies:

$$[N]_{\bar{C}_{nm}\bar{C}_{rs}} = 0 \begin{cases} if\ m \neq s \\ or \\ m = s \ ,\ n - r = 2k+1 \quad k \in \mathbb{N} \end{cases} , \quad (4)$$

for the normal equation element $[N]_{\bar{C}_{nm}\bar{C}_{rs}}$ corresponding to unknowns $\bar{C}_{nm}$ and $\bar{C}_{rs}$. *Pavlis et al.* (1996) had verified that BDLS estimators based on equation (1) can recover *exactly* (within the computer's numerical noise) a set of coefficients, from synthetic ("true"), noiseless 30′×30′ data, provided these data are band-limited, and the Nyquist degree (359) is not exceeded. This is re-confirmed here, with Figure 1 showing that the spectrum of the difference coefficients ("true" - recovered) is within the numerical noise of the computer's arithmetic.

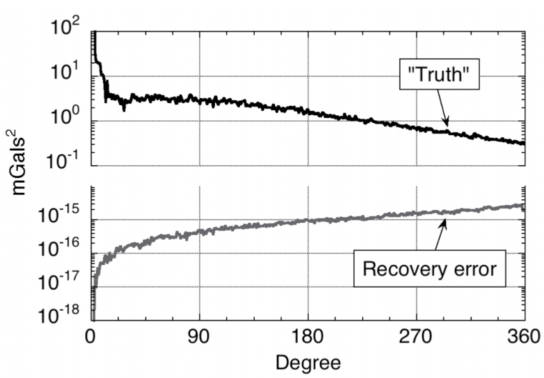

Fig. 1 Spherical harmonic gravity anomaly degree variance and BDLS recovery error (30′×30′ area-mean values on the ellipsoid, N=359).

2 The 5′×5′ Data and Degree 2159 Case

For the new EGM, a corresponding application of the same Type 1 BDLS technique would use 5′×5′ $\overline{\Delta g}^{e}_{ij}$ and form normal equations based on observation equation (1) with N=2159. However, the implementation of this technique within a "closure" experiment using synthetic ("true") 5′×5′ $\overline{\Delta g}^{e}_{ij}$ yielded unexpected results, as shown in Figure 2 (the counterpart of Figure 1, for the 5′×5′ data and N=2159 scenario). From this figure, it is evident that the technique is failing to recover the zonal coefficients, whilst the recovery of the non-zonal coefficients appears to be satisfactory (with the possible exception of the near-Nyquist degrees).

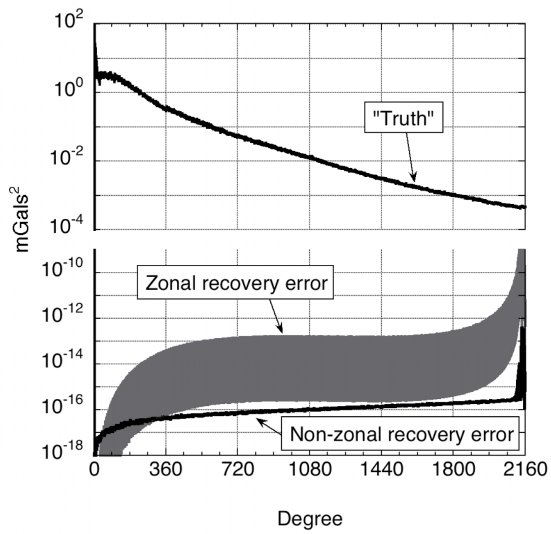

Fig. 2 Spherical harmonic gravity anomaly degree variance and BDLS recovery error (5′×5′ area-mean values on the ellipsoid, N=2159).

Figure 3 shows that the residual gravity anomalies are largest near the poles.

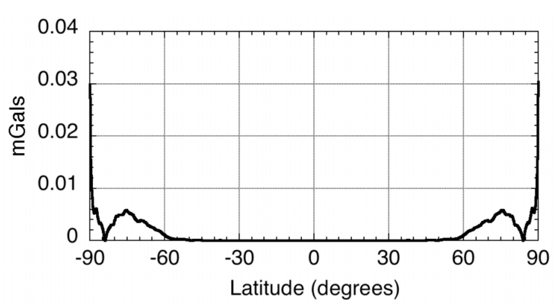

Fig. 3 RMS (per latitude band) residual gravity anomaly (5′×5′ area-mean values on the ellipsoid, N=2159).

Inspection of the normal matrix reveals no reason why the BDLS should fail in this way. As expected, both of the zonal blocks (corresponding to odd and even degrees) are diagonally-dominant. However, Figures 4 and 5 show the zonal blocks of the inverse of the normal equations, normalized such that they represent correlations. For the *N*=359 case, the recovered zonal coefficients are mostly uncorrelated (Figure 4). In stark contrast, for the *N*=2159 case, almost all of the zonal coefficients are almost fully correlated (Figure 5). This suggests the existence of strong dependencies amongst the discretized zonal harmonics within the design matrix, which prevent the proper recovery of zonal harmonic coefficients.

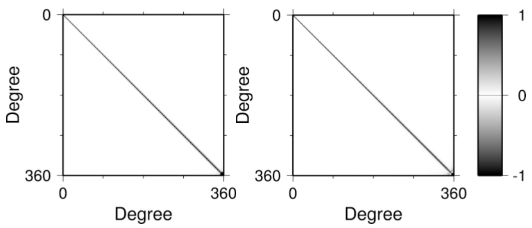

Fig. 4 Correlation matrices for even (left) and odd (right) zonal coefficients (30′×30′ area-mean values on the ellipsoid, *N*=359).

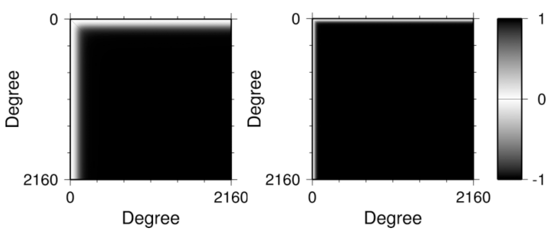

Fig. 5 Correlation matrices for even (left) and odd (right) zonal coefficients (5′×5′ area-mean values on the ellipsoid, *N*=2159).

3 High-Degree Radial Terms

Unlike surface spherical harmonics, the solid spherical harmonics used in equation (1) contain radial $(a/r)^n$ terms, which, on the surface of the ellipsoid, are unity at the equator and increase monotonically towards the poles. For *N*=359 the value of $(a/r)^{359}$ increases to only ~3.3 at the poles. For *N*=2159 the value of $(a/r)^{2159}$ increases to ~1409 at the poles. Legendre polynomials $(\overline{P}_{n0})$, unlike tesseral and sectorial $\overline{P}_{nm}$, contain no $\sin^m\theta$ terms, and so do not taper to zero at the poles. This means, for example, that $(a/r)^{2159}(\overline{P}_{2159,0})$ has an amplitude that is 4 to 5 orders of magnitude larger at the poles than near the equator.

To determine if this extreme variability in the amplitude of the high-degree Legendre polynomials was increasing the non-orthogonality in the disretised zonal harmonics, the 5′×5′, *N*=2159 test closure was repeated for mean gravity anomalies on the surface of the bounding sphere $(r=a)$, instead of on the ellipsoid. For this closure, the $(a/r)^n$ terms are all equal to one, which makes this test an exercise in recovering surface spherical harmonics. As can be seen in Figures 6 and 7, this has removed much of the error in the recovered zonal coefficients, but did not completely eliminate the extreme correlations in the zonal harmonic coefficients.

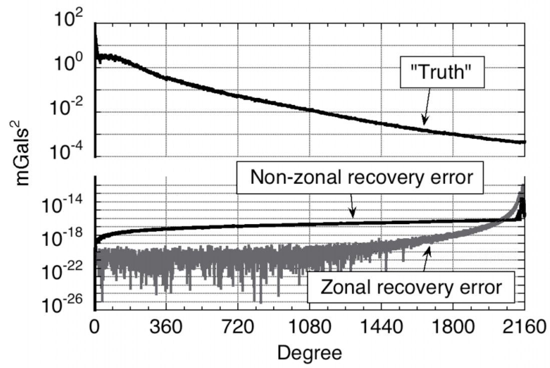

Fig. 6 Spherical harmonic gravity anomaly degree variance and BDLS recovery error (5′×5′ area-mean values on the equatorial bounding sphere *r*=*a*, *N*=2159).

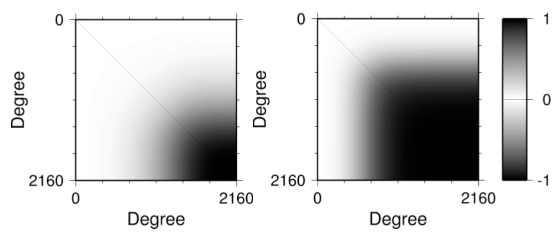

Fig. 7 Correlation matrices for even (left) and odd (right) zonal coefficients (5′×5′ area-mean values on the equatorial bounding sphere *r*=*a*, *N*=2159).

4 Equiangular Geodetic Latitudes

In fact, we must turn to geometrical geodesy to completely remove the correlations in the recovered zonal coefficients. As is common in physical geodesy, the gravity anomalies were averaged over geographic blocks that were equiangular in terms of *geodetic* latitude. Of course, for *spherical* harmonic analysis, this equal spacing in geodetic latitude converts to non-equal spacing in the geocentric latitude that is required as argument for the evaluation of the discretized $\overline{P}_{nm}$. Surprisingly, it is this *specific* latitudinal variation of the geocentric angle contained within each block, although quite smooth and regular, that causes near complete dependencies between the Legendre polynomials ($\overline{P}_{n0}$) for the (5′×5′, *N*=2159) case, but not for the (30′×30′, *N*=359) case. This was verified by repeating, on the surface of the ellipsoid, the (5′×5′, *N*=2159) test closure for anomalies averaged over geographical blocks for which the *geocentric* latitudinal spacing is equiangular. Figure 8 shows the gravity anomaly degree variance for the actual error in the recovered coefficients. Figure 9 shows the correlations in the recovered zonal harmonic coefficients. It is clear that modifying the latitudinal block spacing to geocentric latitude circumvents all the difficulties experienced before in the recovery of *spherical* harmonic coefficients.

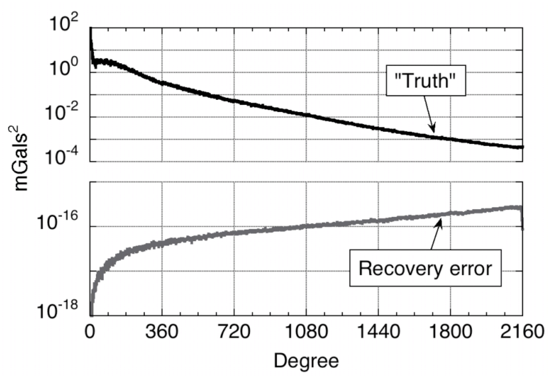

Fig. 8 Spherical harmonic gravity anomaly degree variance and BDLS recovery error (5′×5′ area-mean values on the ellipsoid, equiangular geocentric latitude grid, *N*=2159).

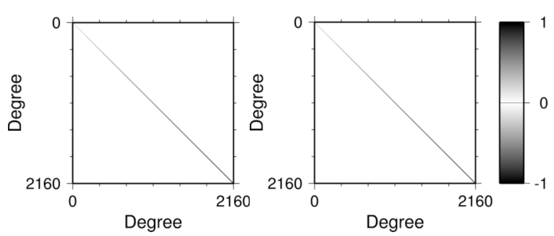

Fig. 9 Correlation matrices for even (left) and odd (right) zonal coefficients (5′×5′ area-mean values on the ellipsoid, equiangular geocentric latitude grid, *N*=2159).

It is certainly possible to estimate gravity anomaly area-means on a 5′×5′ grid that is equiangular in *geocentric* latitude. It is also possible (and most likely preferable) to adhere to the *usual* practice of defining the area-mean gravity anomalies as equiangular in *geodetic* latitude, and to seek another way of resolving the zonal coefficient dependencies. Conceivably this may be accomplished using an *a priori* constraint that could "break down" the correlations amongst zonal spherical harmonic coefficients. Further scrutiny of correlation matrices indicated that high correlations were effectively eliminated when the expansion was restricted to maximum degree 2153 or so (using 5′×5′ data). This indicates that an *a priori* constraint would probably be necessary only for the higher degree zonal coefficients near the Nyquist degree. However, this approach was not tested, since a more appealing alternative is available.

5 Ellipsoidal Harmonics

The preferable alternative involves the implementation of *surface ellipsoidal harmonic analysis*, according to:

$$r_i^e \cdot \overline{\Delta g}_{ij}^{\,e} = \frac{1}{\Delta\sigma_i}\frac{GM}{a}\sum_{n=0}^{N}(n-1)\sum_{m=-n}^{n}\bar{C}^e_{nm}\cdot\overline{IY}^{ij}_{nm} \quad , \qquad (5)$$

rather than the implementation of *solid spherical harmonic analysis*, which is the basis for equation (1). The terms in (5) are defined as in (2) and (3), with the geocentric latitudes being replaced with reduced latitudes. This approach produces the *ellipsoidal* harmonic spectrum of the *harmonic* quantity $r_i^e \cdot \overline{\Delta g}_{ij}^{\,e}$. The corresponding *spherical* harmonic spectrum of this quantity, and hence $\overline{\Delta g}_{ij}^{\,e}$, can be obtained afterwards using *Jekeli's* (1988) transformation. Care should be exercised when comparing ellipsoidal and spherical spectra truncated by degree, since *Jekeli's* (1988) transformation preserves the maximum order, but not the maximum degree.

The ellipsoidal harmonic formulation was verified using a test closure experiment whereby a "true" ellipsoidal harmonic spectrum of $r_i^e \cdot \overline{\Delta g}_{ij}^{\,e}$, complete to *N*=2159, was used to create synthetic 5′×5′ data on the surface of the ellipsoid. Figure 10 shows the gravity anomaly degree variance for the actual error in the recovered ellipsoidal harmonic spectrum. Figure 11 shows the correlation matrices for the zonal ellipsoidal harmonic coefficients. We observe that the conversion of equiangular *geodetic* latitudes to non-equiangular *reduced* latitudes drastically reduced the correlation between zonal ellipsoidal harmonics. Thus the ellipsoidal harmonics formulation offers a viable alternative to the use of *a priori* constraints, at least for 5′×5′ data and up to degree and order 2159.

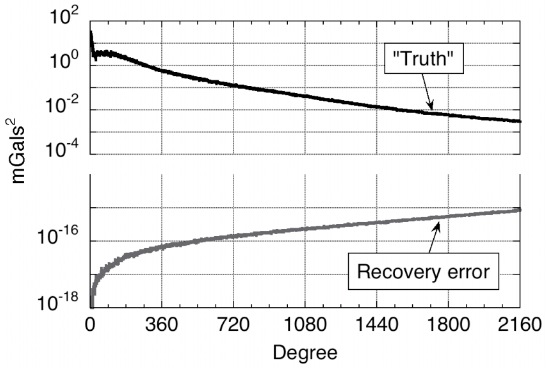

Fig. 10 Ellipsoidal harmonic gravity anomaly degree variance and BDLS recovery error ($5' \times 5'$ area-mean values on the ellipsoid, equiangular geodetic latitude grid, N=2159).

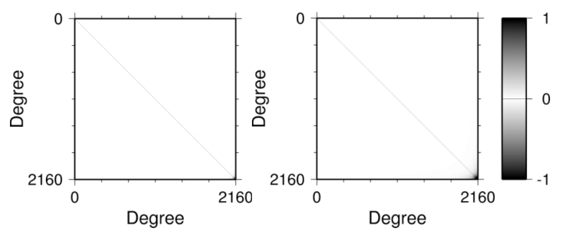

Fig. 11 Correlation matrices for even (left) and odd (right) ellipsoidal harmonic zonal coefficients ($5' \times 5'$ area-mean values on the ellipsoid, equiangular geodetic latitude grid, N=2159).

Jekeli's (1988) transformation implicitly upward continues the ellipsoidal spectrum of a harmonic function defined on the ellipsoid, to the corresponding bounding sphere $r = a$, and it is on this sphere that the solid ellipsoidal harmonics are rigorously transformed to solid spherical harmonics. As *Jekeli* (1988, page 112) points out, Figure 12 illustrates that a function band-limited to degree 2159 in its ellipsoidal harmonic spectrum will contain power, albeit rapidly decreasing with increasing degree, in its spherical harmonic spectrum beyond degree 2159 (and vice versa).

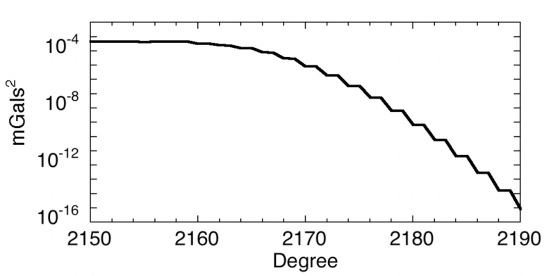

Fig. 12 Spherical harmonic gravity anomaly degree variance obtained using Jekeli's transformation applied on an ellipsoidal spectrum band-limited to N=2159.

If the spherical harmonics computed using Jekeli's transformation are truncated at degree 2159, then the omission error *with respect to the original N=2159 ellipsoidal harmonic spectrum* at the bounding sphere will not be significant (Figure 13).

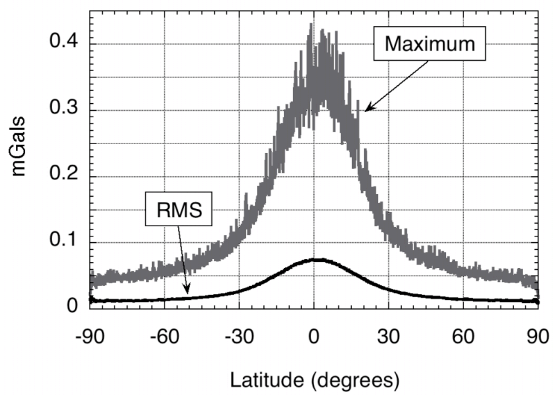

Fig. 13 Gravity anomaly omission error per latitude band at the bounding sphere (r=a), comprising spherical harmonics above degree 2159; i.e., the discrepancy between ellipsoidal harmonic field to degree 2159 and corresponding spherical harmonic field truncated also at degree 2159.

However, when this same degree-2159 truncated spherical harmonic field is downward continued to the ellipsoid, the high-degree $(a/r)^n$ terms greatly amplify the omitted harmonics towards the poles, thereby yielding an omission error on the ellipsoid of over 70 mGal in some points (see Figure 14).

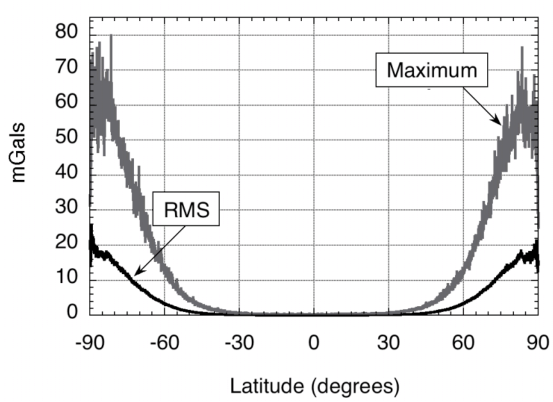

Fig. 14 Gravity anomaly omission error per latitude band, downward continued to the ellipsoid, comprising spherical harmonics above degree 2159.

If the transformed solid spherical harmonic spectrum is truncated at (say) degree 2190, then the omission error on the ellipsoid is everywhere less than 0.0005 mGal (see Figure 15).

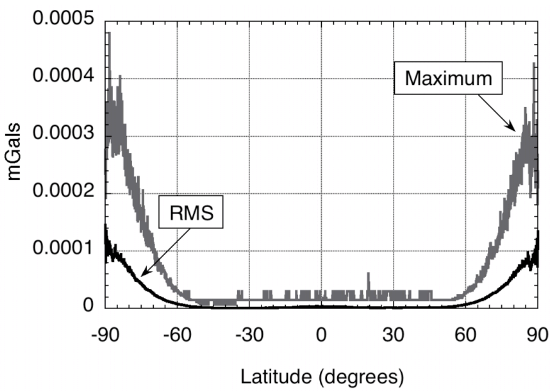

Fig. 15 Gravity anomaly omission error per latitude band, downward continued to the ellipsoid, comprising spherical harmonics above degree 2190.

This amplification of the omitted harmonics above the maximum degree of the ellipsoidal expansion has implications for the spectral filtering of the original data (which aims to produce their band-limited counterparts). In EGM96, the original 30′×30′ gravity anomalies were band-limited by first estimating their ellipsoidal spectrum to N=460. This was then converted to the corresponding spherical spectrum, and then these spherical harmonics above N=359 were removed from the original data. The N=359 band-limited data was then analysed using the solid spherical harmonic formulation of equation (1).

This approach worked for the (30′×30′, N=359) case of EGM96, since the 'mismatch' between spherical and ellipsoidal representations, both to N=359, was not noticeably large at polar latitudes. However, as demonstrated in Figure 14, this technique cannot be used for the present (5′×5′, N=2159) case in the same way, since the omitted spherical harmonics above N=2159 are greatly amplified near the poles. Instead, performing the band-limiting and the "terrestrial" coefficient estimation, all in terms of surface ellipsoidal harmonics as formulated in equation (5), circumvents all these issues. Transforming the satellite-only spectrum from spherical to ellipsoidal harmonics also allows the combination solution with the satellite-only model to be performed in ellipsoidal harmonics. Spherical harmonic coefficients need be computed only for the final combined model, exactly as it was done for the PGM2004A solution developed by *Pavlis et al.* (2005).

6 Summary

In EGM96 spherical harmonic coefficients to degree 359 were computed from 30′×30′ area-mean anomalies using a solid spherical harmonic formulation. For a new EGM, based on 5′×5′ mean anomalies, there are two reasons why this technique cannot be extended directly to degree 2159. First, the conversion of 5′ equiangular *geodetic* latitudes to non-equiangular *geocentric* latitudes results in a near-full correlation of the recovered zonal harmonic coefficients. Second, using ellipsoidal harmonics and Jekeli's transformation to band-limit the 5′×5′ anomalies to *spherical* harmonic degree 2159 can produce a truncated field that does not model the input data adequately near the poles. Both of these problems can be avoided by first band-limiting the 5′×5′ mean anomalies to degree 2159 in the *ellipsoidal* harmonic spectrum, and then by estimating the *surface ellipsoidal* harmonic coefficients to the same degree via block-diagonal least-squares. After combination with a properly transformed satellite-only model, the final (combined) ellipsoidal harmonic spectrum can be converted to the corresponding spherical spectrum using Jekeli's transformation. This spherical spectrum cannot be truncated at degree 2159, since the effects of the omitted spherical harmonics can be amplified to over 70 mGal or so near the poles when downward continued to the ellipsoid. Simply extending the spherical harmonic spectrum to degree 2190 reduces this error to less than 0.5 μGal.

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